Backpaper Exam
Linear Algebra - II
June-2025

Time: 3 hours

Maximum marks: 100

Answer as many questions as you can.

(1) Show that a matrix $A \in M_n(\mathbb{C})$ commutes with every matrix in $M_n(\mathbb{C})$ if and only if there exists an $\alpha \in \mathbb{C}$ such that $A = \alpha I_n$. (15)

(2) Consider the following matrix:

$$A = \begin{pmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

a) Find the eigenvalues of A.
b) Find the eigenvectors of A.
c) Find an orthogonal matrix P, such that $P^{-1}AP$ is diagonal. (15)

(3) Suppose V is of dimension n and $T : V \to V$ is a linear operator. Prove that T is invertible if and only if the identity operator $I \in \text{span}(T, T^2, \ldots, T^n)$. (10)

(4) (a) State and prove spectral decomposition theorem for normal matrices.
(b) Let $A \in M_n(\mathbb{C})$ be a positive semidefinite matrix (i.e. $X^*AX \geq 0$ for all $X \in \mathbb{C}^n$). Use part (a) to show that there exists a positive semidefinite matrix B such that $B^2 = A$. (8+7)

(5) Compute the singular value decomposition (SVD) of the following matrix

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}.$$

(15)

(6) Let $\mathcal{P}_3(\mathbb{R})$ denote the real vector space of all real polynomials of degree atmost 3. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$$

defines an inner product on $\mathcal{P}_3(\mathbb{R})$. Find an orthonormal basis for $\mathcal{P}_3(\mathbb{R})$. (15)

(7) (a) State Sylvester's law of inertia for quadratic forms.
(b) Find the rank, index and signature of the quadratic form given by:

$$Q(x, y, z)^t = x^2 + 2y^2 + 3z^2 + 2xy - 2zx + 2yz.$$

(3+12)

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